

Chapter 2

MATRICES

- 2.1 Solution of Linear Systems
- 2.2 Addition and Subtraction of Matrices
- 2.3 Multiplication of Matrices
- 2.4 Matrix Inverses
- 2.5 Eigenvalues and Eigenvectors

2.1 Solution of Linear Systems

Introduction: An Animal Feed Problem

Suppose three ingredients are available to make an animal feed: corn, soybeans and cottonseed. The table below shows the number of grams of protein, fat and fibre provided by 1 gram of each ingredient.

	corn	soybeans	cottonseed
protein	0.25	0.40	0.20
fat	0.40	0.20	0.30
fibre	0.30	0.20	0.10

Suppose we want to make a feed which contains a total of 22 g of protein, 28g of fat and 18 g of fibre. How much of each ingredient do we need?

Suppose we let x be the required mass of corn (in grams),
let y be the required mass of soybeans, and
let z be the required mass of cottonseed.

Consider the protein needed.

x grams of corn provides $0.25x$ grams of protein.

y grams of soybeans provides $0.40x$ grams of protein.

z grams of corn provides $0.20x$ grams of protein.

So to make a feed which contains a total of 22 g of
protein, we need:

$$0.25x + 0.4y + 0.2z = 22$$

Similarly for fat we need

$$0.4x + 0.2y + 0.3z = 28$$

And for fibre:

$$0.3x + 0.2y + 0.1z = 18$$

We need to find the values of x , y , z which satisfy all three
equations simultaneously. That is, we need to solve this
system of linear equations.

- Many practical problems lead to systems of linear equations.
- Solving a system of 2 or 3 linear equations is fairly easy. But how would we solve a system of 10 or 20 equations?!
- In advanced mathematics, *matrices* are used.
- We will learn matrix methods mainly using examples with just 2 or 3 equations.
- But matrix methods can be applied systematically to any number of equations of any size ... and so can also be used to write computer programs for solving large systems of equations!

Matrices may be considered a topic of algebra, but are also closely related to vectors and geometry.

Linear Systems & Augmented Matrices

Consider a system of n linear equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

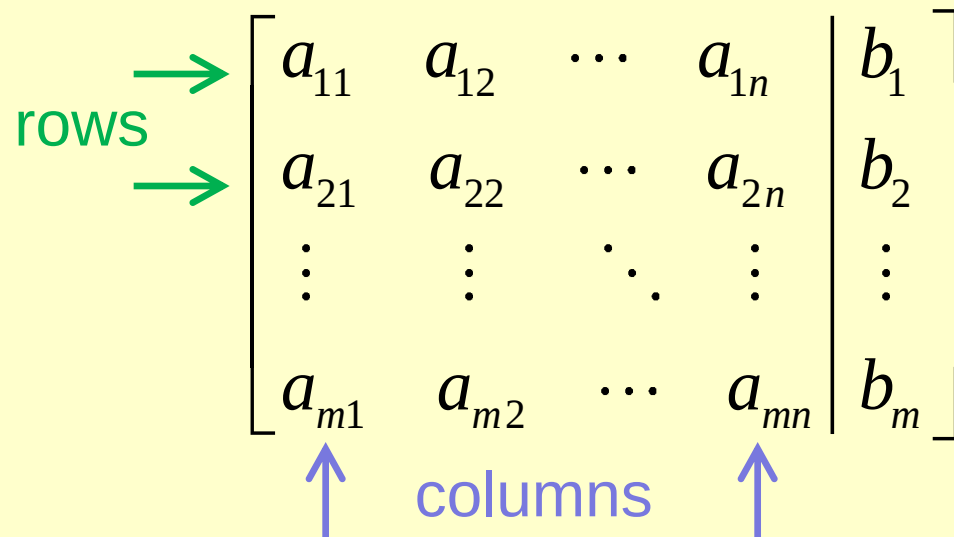
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$\cdots$ *etc.* $\cdots$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where the **coefficients** a_{ij} and the constants b_k are known.

The equations can be written as an **augmented matrix**:



The diagram shows an augmented matrix with m rows and n columns. The matrix is represented as a large square bracket containing a smaller square bracket with coefficients a_{ij} and a vertical line followed by constants b_i . Green arrows point to the rows, and blue arrows point to the columns.

$$\begin{array}{c} \text{rows} \rightarrow \\ \rightarrow \end{array} \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

$\uparrow$ columns $\uparrow$

Each number in the array is called an **element** or **entry**

As an **example** consider the following system of equations:

$$2x \quad \quad -z = 2$$

$$6x + 5y + 3z = 7$$

$$2x - y \quad \quad = 4$$

The augmented matrix is:
$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{array} \right]$$

In the tables on the following slides, the left hand column shows the equations and a way of solving them. At each step, the right hand column shows the corresponding augmented matrix.

We start with

$$2x \quad -z = 2 \quad (1)$$

$$6x + 5y + 3z = 7 \quad (2)$$

$$2x - y \quad = 4 \quad (3)$$

This corresponds to:

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{array} \right]$$

Use (1) to eliminate x from (2),(3):

$$(1): \quad 2x \quad -z = 2$$

$$(2) - 3(1): \quad 5y + 6z = 1$$

$$(3) - (1): \quad -y + z = 2$$

$$\begin{array}{l} r2 - 3r1 \rightarrow r2 \\ \rightarrow \\ r3 - r1 \rightarrow r3 \end{array} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 5 & 6 & 1 \\ 0 & -1 & 1 & 2 \end{array} \right]$$

Interchange the 2nd and 3rd eqns:

$$2x \quad -z = 2$$

$$-y + z = 2$$

$$5y + 6z = 1$$

$$\begin{array}{l} \text{swop } r2, r3 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 5 & 6 & 1 \end{array} \right]$$

Use (2) to eliminate y from (3):

$$(1): \quad 2x \quad -z = 2$$

$$(2): \quad -y + z = 2$$

$$(3) + 5(2): \quad 11z = 11$$

$$r3+5r2 \rightarrow r3 \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 11 & 11 \end{array} \right]$$

Simplify (3):

$$(1) \quad 2x \quad -z = 2$$

$$(2) \quad -y + z = 2$$

$$(3) \div 11 \quad z = 1$$

$$r3 \div 11 \rightarrow r3 \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Use (3) to eliminate z from (1),(2):

$$(1) + (3) \quad 2x \quad = 3$$

$$(2) - (3) \quad -y \quad = 1$$

$$(3) \quad z = 1$$

$$\xrightarrow[r2-r3 \rightarrow r2]{r1+r3 \rightarrow r1} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Simplify: $(1) \div 2 \quad x \quad = 3/2$

$$(2) \times -1 \quad y \quad = -1$$

$$(3) \quad z = 1$$

$$\xrightarrow[-r2 \rightarrow r2]{r1 \div 2 \rightarrow r1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

This is the solution!

The working in the two columns is equivalent – but the matrix form involves less writing!

The operations which we are allowed to perform on the matrix are called *elementary row operations*. They are:

- Interchange two rows
- Multiply or divide a row by a constant
- Add a (positive or negative) multiple of one row to another

Note:

- 1) Augmented matrices related by row operations are equivalent (i.e. have the same solutions) but are *not equal*. You must *not* use “ = ” between the steps. You should *arrows*, and note what operation(s) you do.
- 2) You may do two or more operations in one step provided you do not use a row you are changing as part of another row operation.

- The matrix method demonstrated is called the **Gauss-Jordan method** (or **Gauss elimination** or **row reduction**).
- The numbers in positions a_{11} , a_{22} , etc are called *pivots*. We used each pivot to obtain the zeroes below it.
- A matrix with zeros below the “leading diagonal” is called **echelon form**, e.g.

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 11 & 11 \end{array} \right]$$

leading diagonal

- Finally we get a matrix where the left hand side is an **identity matrix** (see later):

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Then the Gauss- Jordan method is as follows:

1. Write down the *augmented matrix* for the equations.
2. Use row operations to get a suitable *pivot* a_{11} .
Then use a_{11} and row operations to get zeroes *below* a_{11} .
3. Leaving the 1st row and column unchanged, repeat step 2 on the remaining submatrix. I.e. get a suitable pivot a_{22} and use it to get zeroes below it. If there are more than 3 rows, then repeat for pivot a_{33} , a_{44} , etc. until the matrix is in *echelon form*.
4. If possible, use the bottom row to get zeroes above its non-zero element (on left hand side). Repeat for higher rows until all non-diagonal elements are zero. Finally multiply each row by a constant so the left hand side is an *identity* matrix.
5. Read off the solutions.

Example 1

Solve the system of equations

$$x + 5z = y - 6$$

$$3x + 3y = 10 + z$$

$$x + 3y + 2z = 5$$

[Solution: $x = 1$, $y = 2$, $z = -1$.]

Example 2

Solve the system of equations

$$2x - y = 1$$

$$x + 3y = 11$$

Example 3

Solve the system of equations

$$2x_1 + x_2 + x_3 = -2$$

$$-x_2 + 3x_3 = -8$$

$$x_1 + 4x_2 - x_3 = 9$$

[Solution: $x_1 = -1$, $x_2 = 2$, $x_3 = -2$.]

Inconsistent, Dependent and Homogeneous Equations

- So far we have considered systems of n equations in n unknowns and always found a unique, non-zero solution.
- Do such systems always have a unique solution?

Example 4 By sketching the graphs, determine what solutions (if any) the following systems have:

$$\begin{array}{l} a) \ x + y = 2 \\ \quad x - y = 0 \end{array}$$

$$\begin{array}{l} b) \ x + y = 2 \\ \quad 2x + 2y = 5 \end{array}$$

$$\begin{array}{l} c) \ x + y = 2 \\ \quad 2x + 2y = 4 \end{array}$$

$$\begin{array}{l} d) \ x + y = 0 \\ \quad x - y = 0 \end{array}$$

$$\begin{array}{l} e) \ x + y = 0 \\ \quad 2x + 2y = 0 \end{array}$$

Let's see what happens when we use the Gauss-Jordan method on cases (b) – (e).

Example 4(b)

Consider again $x + y = 1$
 $2x + 2y = 5$

The augmented matrix is $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 5 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 3 \end{array} \right]$

- The *bottom row is zero except in the right hand column*
- This indicates that the equations are *inconsistent* and have *no solutions*.

Example 4(c)

Consider again $x + y = 1$
 $2x + 2y = 4$

The augmented matrix is $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 4 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right]$

- We see *the entire bottom row is zero*
- This indicates that the equations are *linearly dependent*.
So we have an infinite number of solutions.

- Normally the bottom row tells us the value of y
- But here y can take any value.
- Then from the top line, $x = 1 - y$.
- So the general solution can be written: $(1 - k, k)$ for any k .

Consider again $d) \begin{cases} x + y = 0 \\ x - y = 0 \end{cases}$ and $e) \begin{cases} x + y = 0 \\ 2x + 2y = 0 \end{cases}$

These systems of equations are called **homogenous** because they contain no constants – the RHSs are zero.

For Example 4(d) the augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

which gives $x = y = 0$. This is called the **trivial solution**.

- In echelon form, *the right hand column is all zeros but the bottom line contains a non-zero entry.*
- This indicates that *the only solution is the trivial solution.*

For Example 4(e) e) $x + y = 0$

$$2x + 2y = 0$$

the augmented matrix is $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$

- In echelon form, again *the entire bottom row is zeroes*.
- This indicates **linear dependence** and **an infinite number of solutions (including the trivial solution)**
 - In this example the solutions are $(k, -k)$ for all k .

Summary: After reducing the matrix to echelon form,

- full row of zeroes $\Rightarrow$ linear dependence (many solutions)
- row of zeroes except for the right hand column
 $\Rightarrow$ inconsistency (no solutions)
- bottom row has non-zero entry and right hand column is all zeroes $\Rightarrow$ have only the trivial solution.

Example 5

Find α such that the equations
solutions.

$$4x + 3y = 0$$

$$2x + \alpha y = 0$$

have many

Find these solutions.

Example 6

Solve (if possible) the system of equations:

$$2x - 2y = 0$$

$$y + z = 4$$

$$x + z = 1$$

What about systems of m equations in n unknowns, $m \neq n$?

If the equations are consistent,

➤ *If $m > n$ the equations will have linear dependence*

➤ *If $m < n$ there will be many solutions*

since exactly n linearly independent equations are needed to find unique solutions.

We can write an augmented matrix and use the Gauss-Jordan method as usual.

(You can explore some examples in the exercises ...)

Example 7

Solve (if possible) the system of equations:

$$x + 2y - z = 5, \quad -2x + z = 8, \quad y + z = 2, \quad -x + y - 2z = -1.$$

Example 8: Application Exercise

A laboratory has a group of 23 rats, including young rats, adult female rats and adult male rats. Each day the group consumes a total of 376 g of food and 56.9 ml of water.

It is known that a young rat consumes 12 g of food, 2 ml of water daily. For an adult female the figures are 16 g and 2.5 ml. For an adult male, 20 g and 2.8 ml. How many young, adult female and adult male rats are there in the group?

2.2 Addition & Subtraction of Matrices

Introduction

A **MATRIX** is a rectangular array of numbers,
e.g.

Square brackets [] or round brackets () may be used.

In general, a matrix A is a set of mn numbers arranged in m **rows** and n **columns**:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} is the **element** (or *entry*) in the i^{th} row and j^{th} column.

A matrix can also be written $A = [a_{ij}]_{m \times n}$

A matrix with m rows and n columns is called an $m \times n$ **matrix** (read: m by n).

It is said to have **ORDER** $m \times n$.

Example

The matrix $A = \begin{bmatrix} 1 & 5 & -4 & 0 \\ 4 & -2 & 3 & 11 \end{bmatrix}$

has order 2×4

and elements $a_{12} = 5$ $a_{24} = 11$

Some special types of matrices:

- If $m = 1$ we have a **ROW MATRIX** (or *row vector*),
e.g.
- If $n = 1$ we have a **COLUMN MATRIX** (*column vector*),
e.g.
- If $m = n$ we have a **SQUARE MATRIX** of order n ,
e.g.
- Elements $a_{11}, a_{22}, \dots, a_{nn}$ are called *diagonal elements*.
- The line on which they lie is called the *leading diagonal*.
- A **DIAGONAL MATRIX** is a square matrix with zeroes everywhere except on the leading diagonal.
E.g.

- A **ZERO MATRIX** (or NULL matrix), O is a matrix all of whose elements are zero.

E.g. the 2×3 zero matrix is

$$O_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- An **IDENTITY MATRIX** (or UNIT matrix), I
 - A square matrix with 1s on the leading diagonal
0s everywhere else.

E.g. the 3×3 identity matrix is

$$I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Equality

Two matrices are equal if and only if

- (i) they have the same order, and
- (ii) corresponding entries are all the same.

I.e. $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are equal if and only if
 $m = p$, $n = q$ and $a_{ij} = b_{ij}$ for all i and all j .

Example

Let $A = \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $A = B$ then $a = 5$, $b = 3$, $c = -2$, $d = 1$.

Addition

- Two matrices can be added if and only if they have the same order.
- The sum is then obtained by adding corresponding elements.

I.e. if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then $A+B = [a_{ij} + b_{ij}]_{m \times n}$

Example 9 Let $A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 10 \\ 5 & -3 \\ 7 & -6 \end{bmatrix}$.

Determine whether the following exist, and if they do exist, find them:

(i) $A + B$

(ii) $A + C$

Scalar Multiplication

Let A be a matrix and k be a scalar.

kA is obtained by multiplying *every* element of A by k .

I.e. If $A = [a_{ij}]$ then $kA = [ka_{ij}]$,

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} .$$

Subtraction

- Writing $A - B = A + (-B)$ and using the above, we see that matrix subtraction can be performed in the same way as addition.
- I.e. Two matrices can be subtracted if and only if they have the same order. Then the difference is obtained by subtracting the corresponding elements.

I.e. if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then $A - B = [a_{ij} - b_{ij}]_{m \times n}$

Example 10 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 4 \\ 2 & 1 & 7 \end{bmatrix}$.

Find

(i) $2B$

(ii) $A + 2B$

(iii) $A - B$

Example 11: Application Exercise

At the beginning of an experiment, four baby rats are measured to have lengths 5.6, 6.4, 7.6 and 6.1 cm and weights 144, 138, 152 and 146 grams, respectively.

Two weeks later the lengths were 10.2, 11.4, 12.7 and 10.8 cm and the weights were 196, 196, 250 and 230 g.

- a) Write the initial data as a 2×4 matrix.
- b) Write the later data as a 2×4 matrix.
- c) Find a matrix representing the changes in length and weight of the rats.

Transpose

For a matrix A , its *transpose*, A^T is obtained by *interchanging* the rows and the columns.

I.e. if $A = [a_{ij}]_{m \times n}$ then $A^T = [a_{ji}]_{n \times m}$.

E.g. for

$$A = \begin{bmatrix} 2 & 0 \\ 6 & 1 \\ 5 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 6 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

- **SYMMETRIC MATRIX** - A matrix is symmetric if $A^T = A$.

This requires that A is square and $a_{ij} = a_{ji}$.

E.g.

2.3 Multiplication of Matrices

Consider a system of n linear equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\cdots \quad \text{etc.} \quad \cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

As an alternative to using an augmented matrix, it would be very convenient to be able to write the system as $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Therefore we use a “**row by column**” definition for multiplication.

E.g. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What is $C = AB$?

$$C = \begin{bmatrix} 1a + 2c & 1b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix} = \begin{bmatrix} \text{Arow1} \times \text{Bcol1} & \text{Arow1} \times \text{Bcol2} \\ \text{Arow2} \times \text{Bcol1} & \text{Arow2} \times \text{Bcol2} \end{bmatrix}$$

I.e. To find c_{ij} - take row i of A and column j of B
- multiply corresponding elements and add

More formally, $c_{ij} = \sum_k a_{ik} b_{kj}$.

E.g. Let $A = \begin{bmatrix} 2 & -3 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 \\ 4 & 6 \end{bmatrix}$. What is $C = AB$?

$$C = \begin{bmatrix} 2.5 - 3.4 & 2.1 - 3.6 \\ -1.5 + 0.4 & -1.1 + 0.6 \end{bmatrix} = \begin{bmatrix} -2 & -16 \\ -5 & -1 \end{bmatrix}$$

Example 12

(a) Let $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$.

Find $C = AB$.

(b) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$.

Find AB and BA

Clearly we can only multiply matrices A and B together if
no. of elements in row of A = no. of elements in column of B

This is equivalent to saying
no. of columns in A = no. of rows in B

If A is a $m \times n$ matrix and

B is a $p \times q$ matrix,

then

- AB exists if and only if $n = p$
- If $n = p$, matrix AB has order $m \times q$.

Example 13 For $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $C = [1 \ 0 \ 8]$,
determine whether the following exist and find those which
exist: (a) AB , (b) BA , (c) BA^T , (d) BC , (e) AC^T .

Note: In general, $AB \neq BA$, i.e. **the order matters!**

There are *special cases* in which $CD = DC$. In such cases we say that matrices C and D **commute**.

Matrix Multiplication: Further Properties

1. Provided the products exist,

$$(AB)C = A(BC) = ABC$$

$$A(B+C) = AB + AC$$

2. For any square matrix A , $AI = IA = A$

$$\text{E.g. } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Also $I^2 = I^k = I$ where k is any positive integer

3. Provided the products exist,

$$(AB)^T = B^T A^T, \quad (ABC)^T = C^T B^T A^T, \text{ etc.}$$

4. $AB = 0$ does NOT necessarily mean $A = 0$ or $B = 0$.

E.g. for $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

5. $AB = AC$ does NOT necessarily mean $B = C$.

E.g. for $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Properties (4) and (5) mean we cannot ‘cancel’ matrices!
Later we will learn about inverse matrices which can sometimes be used instead of cancelling.

Example 14: Application Exercise

A wildlife centre has a bird feeder visited by two types of birds, A and B. The numbers which visit regularly in each season are shown in the matrix P .

$$P = \begin{matrix} & \begin{matrix} \text{A} & \text{B} \end{matrix} \\ \begin{matrix} \text{spring} \\ \text{summer} \\ \text{fall} \\ \text{winter} \end{matrix} & \begin{bmatrix} 25 & 31 \\ 20 & 35 \\ 22 & 29 \\ 36 & 20 \end{bmatrix} \end{matrix}$$

The birds eat different amounts of different foods. Matrix Q shows the mass of food (in grams) that one bird of each species eats over an entire season.

$$Q = \begin{matrix} & \begin{matrix} \text{corn} & \text{millet} \end{matrix} \\ \begin{matrix} \text{A} \\ \text{B} \end{matrix} & \begin{bmatrix} 10 & 30 \\ 18 & 27 \end{bmatrix} \end{matrix}$$

The cost (in cents) per gram of each type of food is shown in matrix R .

$$R = \begin{matrix} & \begin{matrix} \text{corn} \\ \text{millet} \end{matrix} \\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{matrix}$$

- (a) Find $S = PQ$ and explain what it represents.
- (b) What is the total cost of seed for each season?
- (c) How many grams of each kind of seed are used in a year?
- (d) What is the total cost of seed over a year?

2.4 Matrix Inverses

In sections 2.4 & 2.5 we will be concerned only with **square** matrices. We also need the concept of a *determinant*.

Determinants

- A *matrix* is an *array* of numbers. It does not have a single numerical value.
- However for a *square* matrix there is a useful number called the **determinant** of the matrix, denoted $\det A$ or $|A|$.

For a **1 x 1 matrix**, the determinant is simply the value of the single element.

2 × 2 Determinants

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

E.g. For $A = \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$, $|A| = 5 \cdot 2 - 3 \cdot 1 = 10 - 3 = 7$

Example 15

Find the determinants of the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}.$$

3 × 3 Determinants

The general 3 × 3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

has determinant:

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 16 Find the determinants of

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 6 & 3 & 4 \\ 2 & 1 & 5 \end{bmatrix},$$

$$B = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Matrix Inverses

Any scalar $a \neq 0$ has an inverse $a^{-1} = 1/a$ such that

$$a a^{-1} = a^{-1}a = 1.$$

Similarly, a *square* matrix A may have an inverse, A^{-1} , such that

$$AA^{-1} = A^{-1}A = I.$$

A^{-1} exists if and only if $|A| \neq 0$.

- If $|A| = 0$, A is said to be **singular**. It has no inverse.
- If $|A| \neq 0$, A is **non-singular** and has an inverse.

I.e. If $|A| = 0$, there is *no* matrix B such that $AB = I$.

Proof: If $|A| = 0$ then $|A||B| = |AB| = 0$ for any B , meanwhile $|I| = 1 \neq 0$, so we cannot satisfy $AB = I$.

E.g. $C = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$ has $|C| = 0$, so C is singular, C^{-1} does not exist.

Inverse of a 2 x 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has inverse } A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

E.g. for $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, $|A| = 2$, $A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

Example Verify that the formula above gives $A^{-1}A = I$.

Example 17

Find the inverses of the following matrices:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}.$$

Inverses of larger matrices

For 3×3 and larger matrices, formulae for inverses become very complicated. Instead we use a method based on the following logic:

- Matrix multiplication is a row-on-column operation.
- So if we take the identity $AA^{-1} = I$ and apply the *same* row operation to A and I , the equality is maintained.
- Suppose we apply a sequence of row operations that transforms A into I , and transforms I into a new matrix that we call B . The equality will now be $A^{-1} = B$. That is, the new matrix B is the inverse of A .

Method to find A^{-1} for any non-singular square matrix A :

- Form the augmented matrix: $[A \mid I]$
- Perform row operations to get a matrix of form $[I \mid B]$
- Matrix B is A^{-1} .

Example 18

Use the method above to find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

Example 19

Use the method above to find the inverse of $B = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

Example 20 (cf Example 17)

Use the method above to find the inverses of

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 0 & -2 \end{bmatrix}.$$

Note: What happens if we apply this method to a singular matrix?

Example 21

Try to use row operations to find the inverse of $C = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$.

We get a bottom row on the LHS which is all zeroes – so we can go no further. We cannot find an inverse!

Now let's apply this method to a 3×3 matrix.

Example 22 Find the inverse of $D = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

Solving Linear Systems Using Inverse Matrices

Consider again a system of n linear equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\cdots \quad \text{etc.} \quad \cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

As an *alternative* to using an augmented matrix, we can write the system of equations as $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Matrix A is called the **coefficient matrix**.

Consider the equation $AX = B$.

Provided A is non-singular, we can pre-multiply both sides of the equation by A^{-1} :

$$A^{-1}AX = A^{-1}B$$

$$I X = A^{-1}B$$

$$X = A^{-1}B$$

Thus the equations have the unique solution:

$$X = A^{-1}B$$

Note

- Usually the Gauss-Jordan method is quicker!
- But if we have several problems with the same coefficient matrix, or if we already know the inverse, then the inverse matrix method may be quicker.

*For an interesting application to encrypting communication between computers, see Greenwell p.580!

Example 23

Using the inverse of the coefficient matrix, solve the system of equations

$$2x - 3y = 3$$

$$x + 4y = 7$$

Example 24

Solve the given system of equations.

[Hint: see Example 22!]

$$x - y + 2z = -3$$

$$x + 2y + z = 5$$

$$-y + 2z = -1$$

Matrix Inverse approach for Inconsistent, Dependent and Homogeneous Equations

Sometimes equations do not have a unique solution.

Using the matrix inverse method, the nature of the solutions of $AX = B$ depends on

- whether or not A is singular
- whether or not the equations are homogeneous ($B = 0$)

So we have 4 different cases:

- 1) Non-homogeneous, A non-singular [$AX = B$, A^{-1} exists]
Unique solution $X = A^{-1}B$.
- 2) Non-homogeneous, A singular [$AX = B$, A^{-1} does not exist]
No solutions (inconsistent)
or Infinitely many solutions (linearly dependent)
- 3) Homogeneous, A non-singular [$AX = 0$, A^{-1} exists]
Only have trivial solution, $X = 0$.
- 4) Homogeneous, A singular [$AX = 0$, A^{-1} does not exist]
Infinitely many solutions (including the trivial).

Note:

- Unique solution only if $|A| \neq 0$ (cases 1 & 3).
- Infinitely many solutions only if $|A| = 0$ (cases 2 & 4).
- $AX = 0$ has non-trivial solution iff $|A| = 0$ (compare 3, 4)

Example 25

Find α such that the system of equations shown has non-trivial solutions. Find these solutions.

$$x + 5y + 3z = 0$$

$$5x + y - \alpha z = 0$$

$$x + 2y + \alpha z = 0$$

Outline Solution

Equations are homogeneous so for non-trivial solutions require $|A| = 0$. We find $|A| = -27\alpha + 27$ so need $\alpha = 1$.

Set $\alpha = 1$. We know the equations are linearly dependent. Working directly from two of the equations and setting $x = k$ (or using an augmented matrix) we obtain solutions:

$$x = k, \quad y = -2k, \quad z = 3k \quad \text{for all } k.$$

2.5 Eigenvalues and Eigenvectors

Example 26

For $A = \begin{bmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \text{find } AX.$

We can see that the answer is a scalar multiple of the original column vector X . We can say that X is an **eigenvector** of matrix A with **eigenvalue** 4.

Theory

If $AX = \lambda X$ where A is a square matrix,
 X is a column matrix, $X \neq 0$,
 λ is a scalar,
then λ is called an **eigenvalue** of the matrix A ,
and X is called an **eigenvector** corresponding to the
eigenvalue λ .

Note: If X is an eigenvector then kX is also an eigenvector,
for any scalar k . I.e. Eigenvectors are not unique.

Finding Eigenvalues and Eigenvectors

The equation $AX = \lambda X$ (1)

can be written: $AX = \lambda IX$

$$(A - \lambda I)X = 0 \quad (2)$$

This is a homogeneous equation so has non-trivial solutions if and only if $|A - \lambda I| = 0$ (3)

Equation (3) is called the **characteristic equation**.

$\Rightarrow$ To find the eigenvalues, solve the characteristic equation to find λ .

If A has order n then $|A - \lambda I|$ is a polynomial of degree n in λ so solving (3) yields n eigenvalues.

$\Rightarrow$ To find the eigenvector corresponding to a given eigenvalue λ , substitute λ back in to equation (2) and solve for X .

Example 27

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$

Example 28

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & 2 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

Example 29

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 5 & 2 & 3 \\ 0 & 8 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

Application to Populations

Problem: How can we determine a population of insects that grows at a constant rate while keeping the same proportion of adults to juveniles?

Suppose certain insects live a maximum of two years.

Let $x_1(t)$ be the number of juveniles (age < 1 year) in year t .

Let $x_2(t)$ be the number of adults ($1 \leq \text{age} < 2$) in year t .

We also define:

s : proportion of juveniles that survive to be adults the next year.

p : proportion of juveniles that have offspring that are juveniles the next year.

q : proportion of adults that have offspring that are juveniles the next year.

The populations the next year will be:

$$\begin{aligned}x_1(t+1) &= px_1(t) + qx_2(t) \\x_2(t+1) &= sx_1(t)\end{aligned}$$

In matrix form, $X(t+1) = M X(t)$
where

$$M = \begin{bmatrix} p & q \\ s & 0 \end{bmatrix}, \quad X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Matrix M is known as a **Leslie matrix**. (For more complicated population problems we can write similar but larger matrices.)

Example 30 Suppose $s = 0.9$, $p = 0.6$, $q = 0.8$.

(a) Write down the Leslie matrix, M .

(b) If $X(0) = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$, find $X(1)$ and $X(2)$.

(c) If $X(0) = \begin{bmatrix} 100 \\ 75 \end{bmatrix}$, find $X(1)$.

(d) Comment on your answers to (b) and (c).

(e) Find the eigenvalues (λ_1, λ_2) and eigenvectors (V_1, V_2) of M .

Notes

1. As this example demonstrates, the *eigenvectors* of a Leslie matrix are populations that grow steadily maintaining a constant ratio of adults to juveniles.
2. Some eigenvalues correspond to population growth, some to decline. (But note that in practice we cannot have negative populations!)
3. It can be helpful to write other populations as linear combinations of eigenvectors. Then we can see more easily how the population will grow/decline.
4. After a long time, any population approaches a multiple of an eigenvector (see Greenwell p592-593).

Example 30, cont.

f) If $X(0) = \begin{bmatrix} 100 \\ 75 \end{bmatrix}$, write down a formula for $X(n)$.

g) If $X(0) = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$, write this as a linear combination of the eigenvectors of M .

h) Use your answer to (g) to recalculate $X(1)$ in this case, and find a formula for $X(n)$. What happens to this population after a long time?